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LUNAR PERTURBATIONS ON ARTIFICIAL SATELLITES OF THE EARTH

Giorgio E. O. Giacaglia

Smithsonian Astrophysical Observatory

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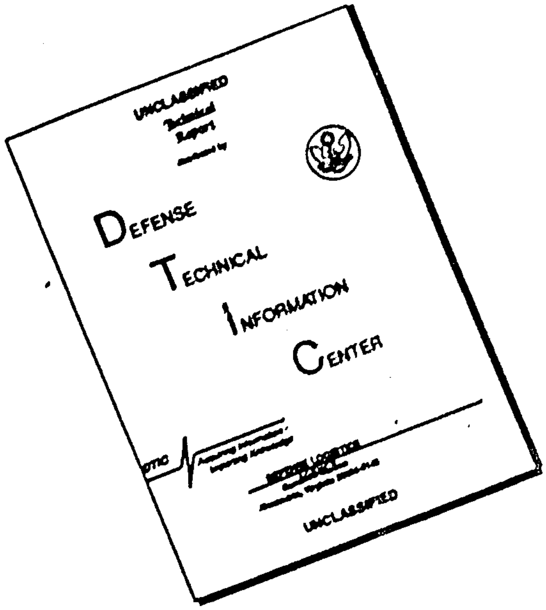

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ABSTRACT

Lunisolar perturbations for general terms of the disturbing function were derived by Kaula (1962). However, his formulas use equatorial elements for the Moon and do not give a definite algorithm for computational procedures. As Kozai (1966) suggested, both inclination and node of the Moon's orbit with respect to the equator of the Earth are not simple functions of time, while the same elements with respect to the ecliptic are well approximated by a constant and a linear function of time, respectively. In the present work, we obtain the disturbing function for the Moon's perturbations using ecliptic elements for the Moon and equatorial elements for the satellite. Secular, long-period, and short-period perturbations are then computed, with the expressions kept in closed form in both inclination and eccentricity of the satellite. Alternative expressions for short-period perturbations of high satellites are also given, assuming small values of the eccentricity. The Moon's position is specified by the inclination, node, argument of perigee, true (or mean) longitude, and its radius vector from the center of the Earth. We can then apply the results to numerical integration by using coordinates of the Moon from ephemeris tapes or to analytical representation by using results from lunar theory, with the Moon's motion represented by a precessing and rotating elliptical orbit.

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RESUME

Kaula (1962) a déduit les perturbations lunisolaires pour les termes généraux de la fonction de perturbation. Toutefois, ses formules emploient des éléments équatoriaux pour la lune et ne donnent pas un algorithme défini pour les procédés de calcul. Comme l'a suggéré Kozai (1966), l'inclinaison et le noeud de l'orbite lunaire ne sont pas de simples fonctions du temps par rapport à l'orbite terrestre, tandis que ces mêmes éléments peuvent être assimilés, respectivement, à une constante et à une fonction linéaire du temps par rapport à l'écliptique. Dans cette étude nous avons obtenu la fonction de perturbation pour les perturbations lunaires en employant des élémerts écliptiques pour la lune et des éléments équatoriaux pour le satellite. Nous avons alors calculé les perturbations séculaires et les perturbations à longues et courtes périodes, les expressions de l'inclinaison et de l'eccentricité du satellite restant délimitées. Nous donnons aussi des expressions alternatives pour des perturbations à courtes périodes de hauts satellites, en supposant que les valeurs de l'eccentricité sont petites. La position de la lune est donnée par l'inclinaison, le noeud, l'argument du périgée, la longitude vraie (ou moyenne), et son vecteur à partir du centre de la terre. Nous pouvons ensuite appliquer les résultats soit à une intégration numérique, en nous servant des coordonnées de la lune données par les éphémérides, soit à une représentation analytique en nous servant des résultats de la théorie lunaire, le mouvement de la lune étant représenté par une orbite elliptique rotatoire avec précession.

КОНСПЕКТ

Значения лунно-солнечных возмущений для общих членов возмущающей функции были выведены Каула (1962). Однако в его формулах используются экваториальные элементы для луны и не дается определенного алгоритма для проведения вычислений. Согласно указаниям Козаи (1966), как наклонение, так и узел лунной орбиты относительно земного экватора не являются простыми функциями времени, в то время как относительно эклиптики те же самые элементы хорошо аппроксимированы соответственно константной и линейной функцией времени. В данной работе мы получаем возмущающую функцию для лунных возмущений, используя элементы эклиптики для луны и экваториальные элементы для спутника. Затем вычисляются вековые, долговременные и кратковременные возмущения, причем и в наклонении и в эксцентриситете спутника выражения даны в виде замкнутых формул. В работе также приведены альтернативные выражения для кратковременных возмущений спутников, находящихся на высокой орбите, причем сделаны допущения о том, что эксцентриситет имеет малые значения. Положение луны определяется наклонением, узлом, аргументом перигея, истинной (или средней) долготой и радиусом-вектором луны от центра земли. Затем полученные результаты мы можем применить для численного интегрирования, используя координаты луны из записей эфемерид на лентах, или для аналитического моделирования, используя результаты лунной теории, причем движение луны представлено в виде прецессирующей и вращающейся эллиптической орбиты.

LUNAR PERTURBATIONS ON
ARTIFICIAL SATELLITES OF THE EARTH

Giorgio E. O. Giacaglia

1. ELEMENTS FOR THE MOON AND OTHER QUANTITIES

Let T be the time in centuries of 36525 ephemeris days from J.D. 2415020.0. The following values will be adopted:

Eccentricity of the Moon:

$$e_☾ = 0.054900489 \quad .$$

Inclination of the Moon:

$$I_☾ = 5°8'43''.427$$

$$\sin (I_☾/2) = 0.044886967 \quad .$$

Mass ratio, Moon to Earth:

$$m_☾/m_\oplus = 0.0123001 \quad .$$

Mean equatorial parallax of the Moon:

$$p_☾ = 57'2''.70 \quad ,$$

where

$$p_☾ = \arcsin (a_e/a_☾) \quad .$$

Mean equatorial radius of the Earth:

$$a_e = 6378160 \text{ m} \quad .$$

$a_☾$ = perturbed semimajor axis of the Moon's orbit.

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Mean motion in longitude of the Moon:

$$n_{\text{☾}} = 13^\circ 10' 34''.889902 \text{ day}^{-1} \quad .$$

Mean anomaly of the Moon:

$$M_{\text{☾}} = 36^\circ 55' 16''.80 + 1724878768''.03\ T + 25''.61\ T^2 + 0''.0438\ T^3 \quad .$$

Argument of Moon's perigee from ecliptic node:

$$\omega_{\text{☾}} = -25^\circ 40' 13''.60 + 14648522''.51\ T - 37''.17\ T^2 - 0''.0450\ T^3 \quad .$$

Ecliptic longitude of Moon's ascending node:

$$\Omega_{\text{☾}} = 259^\circ 10' 59''.79 - 6962911''.23\ T + 7''.48\ T^2 + 0''.0080\ T^3 \quad .$$

Obliquity of the ecliptic:

$$\varepsilon = 23^\circ 27' 08''.26 - 46''.845\ T - 0''.0059\ T^2 + 0''.00181\ T^3 \quad .$$

Explicit and precise variations of ε in terms of the elements of the Moon and the Sun are also available:

$$\varepsilon = \varepsilon_0 + \Omega + d\omega$$

(e.g., Connaissance des Temps, 1971).

2. THE DISTURBING FUNCTION

The disturbing function can be written as

$$R = \frac{Gm_☾}{r_☾} \sum_{\ell \geq 2} \left(\frac{r}{r_☾}\right)^{\ell} P_{\ell} (\cos \psi') \quad , \tag{1}$$

where ψ' is the geocentric elongation of the satellite from the Moon.

The following size considerations apply:

$$Gm_☾ = \frac{m_☾}{m_☾ + m_\oplus} n_☾^2 a_☾^3 \simeq 0.0123\, n_☾^2 a_☾^3 \quad ,$$

also written as

$$Gm_☾ = N_☾^2 a_☾^3 \quad , \qquad N_☾^2 \simeq 1.59 \times 10^{-5} \text{ rev}^2 \text{ day}^{-2} \quad .$$

The satellite Keplerian negative energy is

$$F_0 = n^2 a^2/2 \quad ,$$

so that the relative size of the perturbing force function is given by

$$\nu = R/F_0 = 2\, N_☾^2/n^2 \quad .$$

For low satellites ($T \simeq 90$ min), $\nu \simeq 1.2 \times 10^{-7}$. For high satellites ($T \simeq 24$ h), $\nu \simeq 3.18 \times 10^{-5}$. It follows that, in the above range of periods, for moderate eccentricities, the dominant part of the disturbing function of a satellite is due to the Earth oblateness (J_2), and lunar perturbations are about second order with respect to this.

Let α, δ and α', δ' be the right ascension and declination of the satellite and of the Moon, respectively (in an equatorial system). It follows that

$$\cos \psi' = \cos \delta \cos \delta' \cos (\alpha - \alpha') + \sin \delta \sin \delta' \quad .$$

Therefore,

$$R = \sum_{\ell \geq 2} N_{☾}^{2} a_{☾}^{2-\ell} R_{\ell} \quad , \tag{2}$$

where

$$R_{\ell} = a^{\ell} \left(\frac{r}{a}\right)^{\ell} \left(\frac{a_{☾}}{r_{☾}}\right)^{\ell+1} P_{\ell} (\cos \psi') \quad , \tag{3}$$

or, making use of Legendre's addition theorem,

$$R_{\ell} = a^{\ell} \left(\frac{r}{a}\right)^{\ell} \left(\frac{a_{☾}}{r_{☾}}\right)^{\ell+1} \sum_{m=0}^{\ell} \epsilon_{m} \frac{(\ell - m)!}{(\ell + m)!} P_{\ell}^{m} (\sin \delta') P_{\ell}^{m} (\sin \delta) \cos m(\alpha - \alpha') \quad , \tag{4}$$

where

$$\epsilon_{m} = \begin{cases} 1 \ , & m = 0 \\ 2 \ , & m \neq 0 \ . \end{cases}$$

Let

$$A_{\ell}^{m} = \left(\frac{a_{☾}}{r_{☾}}\right)^{\ell+1} \frac{(\ell - m)!}{(\ell + m)!} \epsilon_{m} P_{\ell}^{m} (\sin \delta') \cos m\alpha' \quad ,$$

$$B_{\ell}^{m} = \left(\frac{a_{☾}}{r_{☾}}\right)^{\ell+1} \frac{(\ell - m)!}{(\ell + m)!} \epsilon_{m} P_{\ell}^{m} (\sin \delta') \sin m\alpha' \quad . \tag{5}$$

We can write

$$R_\ell = a^\ell \left(\frac{r}{a}\right)^\ell \sum_{m=0}^{\ell} (A_\ell^m \cos m\alpha + B_\ell^m \sin m\alpha)\, P_\ell^m (\sin \delta) \quad . \tag{6}$$

Let

$v = \omega + f =$ argument of latitude of satellite,

$\Omega =$ longitude (equatorial) of node of satellite,

$I =$ inclination of satellite to Earth equator.

By considering the relations

$$\begin{aligned} \cos(\alpha - \Omega)\cos\delta &= \cos v \quad , \\ \sin(\alpha - \Omega)\cos\delta &= \sin v \cos I \quad , \\ \sin\delta &= \sin v \sin I \quad , \end{aligned} \tag{7}$$

it follows that

$$\begin{aligned} R_\ell = a^\ell \left(\frac{r}{a}\right)^\ell \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} F_{\ell mp}(I) \Bigg\{ &\begin{bmatrix} A_\ell^m \\ -B_\ell^m \end{bmatrix}_{\ell - m \text{ odd}}^{\ell - m \text{ even}} \cos[(\ell - 2p)v + m\Omega] + \\ &+ \begin{bmatrix} B_\ell^m \\ A_\ell^m \end{bmatrix}_{\ell - m \text{ odd}}^{\ell - m \text{ even}} \sin[(\ell - 2p)v + m\Omega] \Bigg\} \quad , \end{aligned} \tag{8}$$

where (Kaula, 1961)

$$\begin{aligned} F_{\ell mp}(I) = &\sum_i \frac{(2\ell - 2i)!\; 2^{2i - 2\ell}}{i!\,(\ell - i)!\,(\ell - m - 2i)!} \sin^{\ell - m - 2i} I \times \\ &\times \sum_j \binom{m}{j} \cos^j I \sum_k \binom{\ell - m - 2i + j}{k} \binom{m - j}{p - i - k} (-1)^{k - q} \end{aligned} \tag{9}$$

and

$q = \left[\frac{\ell - m}{2}\right]$, the integral part of $(\ell - m)/2$,

$i = 0, 1, 2, \ldots,$ min (p, q),

$j = 0, 1, 2, \ldots, m$,

$k =$ all values for which the coefficient is not zero; that is, $p - i \geq k$.

3. ROTATION OF SPHERICAL HARMONICS FOR THE MOON

The spherical harmonics $P_\ell^m(\sin\delta')\,e^{i m\alpha'}$ are expressed in terms of $\delta_☾$ and $\alpha_☾$, the Moon's ecliptic latitude and longitude.

The relations are given by

$$\cos\delta'\,e^{i\alpha'} = \cos\delta_☾\cos\alpha_☾ + i(\cos\delta_☾\sin\alpha_☾\cos\mathcal{E} - \sin\delta_☾\sin\mathcal{E}) \quad ,$$

$$\sin\delta' = \cos\delta_☾\sin\alpha_☾\sin\mathcal{E} + \sin\delta_☾\cos\mathcal{E} \quad . \tag{10}$$

From the well-known properties of spherical harmonics under rotation, we can write

$$P_\ell^m(\sin\delta')\,e^{i m\alpha'} = \sum_{r=-\ell}^{\ell} a_\ell^{m,r}\,P_\ell^r(\sin\delta_☾)\,e^{i r\alpha_☾} \quad ,$$

where $a_\ell^{m,r}$ is a function of $\mathcal{E}$ only, and

$$P_\ell^{-m}(x) = (-1)^m\,\frac{(\ell-m)!}{(\ell+m)!}\,P_\ell^m(x) \quad .$$

Using the orthogonality conditions of spherical harmonics, we have

$$a_\ell^{m,r} = \frac{\epsilon_r(2\ell+1)}{4\pi}\,\frac{(\ell-r)!}{(\ell+r)!}\int_{-\pi/2}^{+\pi/2}\cos\delta_☾\,d\delta_☾\,\times$$

$$\times\int_0^{2\pi} P_\ell^m(\sin\delta')\,e^{i m\alpha'}\,P_\ell^r(\sin\delta_☾)\,e^{i r\alpha_☾}\,d\alpha_☾ \quad .$$

Introducing Equations (10) and computing the above integral, we find (see also Lee, 1971)

$$a_\ell^{m,r} = \frac{(\ell - r)!}{(\ell - m)!} \, e^{i(m-r)\pi/2} \, U_\ell^{m,r} \quad , \tag{11}$$

where, for $m + r \geq 0$,

$$U_\ell^{m,r} = (-1)^{\ell - m} \binom{\ell + m}{\ell - r} \left(\cos \frac{\varepsilon}{2}\right)^{m+r} \left(\sin \frac{\varepsilon}{2}\right)^{r-m} \times$$

$$\times F\left(-\ell + r,\ \ell + r + 1,\ m + r + 1;\ \cos^2 \frac{\varepsilon}{2}\right) \tag{12a}$$

and, for $m + r \leq 0$,

$$U_\ell^{m,r} = (-1)^{\ell - r} \binom{\ell - m}{\ell + r} \left(\cos \frac{\varepsilon}{2}\right)^{-m-r} \left(\sin \frac{\varepsilon}{2}\right)^{m-r} \times$$

$$\times F\left(-\ell - r,\ \ell - r + 1,\ -m - r + 1;\ \cos^2 \frac{\varepsilon}{2}\right) \ . \tag{12b}$$

In the above relations, F is the usual hypergeometric series ${}_1F_2$, defined by

$$F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad ,$$

where

$$(a)_n = a(a+1)\,(a+2)\,\ldots\,(a+n-1) \quad ,$$

$$(a)_0 = 1 \quad .$$

In both cases, $U_\ell^{m,r}$ is a polynomial in $\sin \varepsilon/2$, $\cos \varepsilon/2$ since at least one of the parameters a, b is negative, and the above series terminate. The distinction of the cases $m + r \gtrless 0$ is necessary to avoid a singularity in F due to a negative value of c. Considering this fact, another possible form for $U_\ell^{m,r}$, valid in any case, is found to be

$$U_\ell^{m,r} = \frac{(-1)^{m-r}}{(\ell+r)!}\left(\cos\frac{\varepsilon}{2}\right)^{m+r}\left(\sin\frac{\varepsilon}{2}\right)^{r-m}\frac{d^{\ell+r}}{dz^{\ell+r}}\left[z^{\ell-m}(z-1)^{\ell+m}\right] \quad , \quad (13)$$

where $z = \cos^2(\varepsilon/2)$.

Now, let

$$2\,A_\ell^{m,r} = U_\ell^{m,r} + (-1)^r\, U_\ell^{m,-r} \quad ,$$

$$2\,B_\ell^{m,r} = U_\ell^{m,r} - (-1)^r\, U_\ell^{m,-r} \quad . \qquad (14)$$

It follows that, for m even,

$$A_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1}\frac{(-1)^m\,\epsilon_m}{(\ell+m)!}\sum_{r=0}^{\ell}(\ell-r)!\,\epsilon_r A_\ell^{m,r}\,P_\ell^r(\sin\delta_☾)\cos\left[r\left(\alpha_☾+\frac{\pi}{2}\right)\right] \quad ,$$

$$B_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1}\frac{(-1)^m\,\epsilon_m}{(\ell+m)!}\sum_{r=1}^{\ell}(\ell-r)!\,\epsilon_r B_\ell^{m,r}\,P_\ell^r(\sin\delta_☾)\sin\left[r\left(\alpha_☾+\frac{\pi}{2}\right)\right] \quad ,$$

(15a)

and, for m odd,

$$A_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1}\frac{(-1)^{m+1}\,\epsilon_m}{(\ell+m)!}\sum_{r=0}^{\ell}(\ell-r)!\,\epsilon_r B_\ell^{m,r}\,P_\ell^r(\sin\delta_☾)\sin\left[r\left(\alpha_☾-\frac{\pi}{2}\right)\right] \quad ,$$

$$B_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1}\frac{(-1)^m\,\epsilon_m}{(\ell+m)!}\sum_{r=0}^{\ell}(\ell-r)!\,\epsilon_r A_\ell^{m,r}\,P_\ell^r(\sin\delta_☾)\cos\left[r\left(\alpha_☾-\frac{\pi}{2}\right)\right] \quad .$$

(15b)

Making use of relations (7) for the Moon, we find that, for m even:

$$A_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1} \frac{(-1)^m \epsilon_m}{(\ell+m)!} \sum_{r=0}^{\ell} (\ell - r)!\, \epsilon_r A_\ell^{m,r} \sum_{p=0}^{\ell} F_{\ell rp}(I_☾) \cos \theta'_{\ell pr} \quad ,$$

$$B_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1} \frac{(-1)^m \epsilon_m}{(\ell+m)!} \sum_{r=0}^{\ell} (\ell - r)!\, \epsilon_r B_\ell^{m,r} \sum_{p=0}^{\ell} F_{\ell rp}(I_☾) \sin \theta'_{\ell pr} \quad ,$$

(16)

and, for m odd:

$$A_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1} \frac{(-1)^{m+1} \epsilon_m}{(\ell+m)!} \sum_{r=0}^{\ell} (\ell - r)!\, \epsilon_r B_\ell^{m,r} \sum_{p=0}^{\ell} F_{\ell rp}(I_☾) \sin \theta'_{\ell pr} \quad ,$$

$$B_\ell^m = \left(\frac{a_☾}{r_☾}\right)^{\ell+1} \frac{(-1)^m \epsilon_m}{\ell+m)!} \sum_{r=0}^{\ell} (\ell - r)!\, \epsilon_r A_\ell^{m,r} \sum_{p=0}^{\ell} F_{\ell rp}(I_☾) \cos \theta'_{\ell pr} \quad ,$$

(17)

where

$$\theta'_{\ell pr} = (\ell - 2p)\, v_☾ + r\left(\Omega_☾ + \frac{\pi}{2}\right) \tag{18}$$

and the functions $F_{\ell rp}(I_☾)$ are defined by Equation (9).

Finally, the function R_ℓ can be written, with Equations (8), (14), (16), and (17) taken into account:

$$R_\ell = a^\ell \left(\frac{r}{a}\right)^\ell \left(\frac{a_☾}{r_☾}\right)^{\ell+1} \sum_{m=0}^{\ell} \sum_{s=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=0}^{\ell} \frac{(-1)^m \epsilon_m \epsilon_s (\ell - s)!}{(\ell+m)!} F_{\ell mp}(I)\, F_{\ell sq}(I_☾)$$

$$\times \left[(-1)^{\ell+m-s} U_\ell^{m,-s} \cos(\theta_{\ell pm} + \theta'_{\ell qs}) + U_\ell^{m,s} \cos(\theta_{\ell pm} - \theta'_{\ell qs})\right] \quad , \tag{19}$$

where

$$\theta_{\ell pm} = (\ell - 2p)\, v + m\Omega \quad . \tag{20}$$

4. SECULAR AND LONG-PERIOD TERMS OF THE DISTURBING FUNCTION

If we assume that no resonance occurs between the orbital motion of the satellite and that of the Moon – that is, $pn + p'n_{☾}$ is not small for small intergers p, p' not simultaneously zero – then the elimination of short-period terms (depending on the mean anomaly of the satellite, M) from the disturbing function can be obtained by making use of the well-known integrals

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\ell}\sin(\ell-2p)f\,dM = 0 \quad ,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\ell}\cos(\ell-2p)f\,dM = (1+\beta^2)^{-\ell-1}\,X_{0,0}^{\ell,\ell-2p}(\beta) = (1+\beta^2)^{-\ell-1}\,H_{\ell p(2p-\ell)}(\beta) \ . \tag{21}$$

In the above relations,

$$\beta = e\left(1+\sqrt{1-e^2}\right)^{-1} \tag{22}$$

and the X's are Hansen's coefficients (e.g., see Plummer, 1960, p. 45) and the H's are Kaula's coefficients (Kaula, 1961). They are defined by, for $2p-\ell > 0$,

$$H_{\ell p(2p-\ell)} = (-\beta)^{2p-\ell}\binom{2p+1}{2p-\ell}F(-\ell-1,\ 2p-2\ell-1,\ 2p-\ell;\ \beta^2) \tag{23}$$

and, for $2p-\ell \leq 0$,

$$H_{\ell p(2p-\ell)} = (-\beta)^{\ell-2p}\binom{2\ell-2p+1}{\ell-2p}F(-\ell-1,\ -2p-1,\ \ell-2p+1;\ \beta^2) \ . \tag{24}$$

In both cases, they are polynomials in β. The distinction again is necessary in order to avoid singular representation.

The long-period and secular part of the function R_ℓ is then found to be

$$\bar{R}_\ell = a^\ell \left(\frac{a_☾}{r_☾}\right)^{\ell+1} \sum_{m=0}^{\ell} \sum_{s=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=0}^{\ell} \frac{(-1)^m \epsilon_m \epsilon_s (\ell - s)!}{(\ell + m)!} F_{\ell mp}(I) F_{\ell sq}(I_☾) \times$$

$$\times (1+\beta^2)^{-\ell-1} H_{\ell p(2p-\ell)}(\beta) \left[(-1)^{\ell+m-s} U_\ell^{m,-s} \cos(\bar{\theta}_{\ell pm} + \theta'_{\ell qs}) + \right.$$

$$\left. + U_\ell^{m,s} \cos(\bar{\theta}_{\ell pm} - \theta'_{\ell qs})\right] \quad , \tag{25}$$

where

$$\bar{\theta}_{\ell pm} = (\ell - 2p)\,\omega + m\Omega \quad . \tag{26}$$

5. SECULAR AND LONG-PERIOD LINEAR PERTURBATIONS

The Lagrange planetary equations can be written as

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M} ,$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega} ,$$

$$\frac{dI}{dt} = \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} - \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega} ,$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} ,$$

$$\frac{d\omega}{dt} = - \frac{\cot I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} ,$$

$$\frac{d\Omega}{dt} = \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial I} , \tag{27}$$

where $n = \mu^{1/2}\, a^{-3/2}$ if it appears outside trigonometric functions and

$$M = \sigma + \int n\, dt = \sigma - 3\int\left(\int \frac{1}{a^2}\frac{\partial R}{\partial M}\, dt\right) dt . \tag{28}$$

In the above relation, σ contains all perturbations defined in the fourth equation of (27). The last term of Equation (28) contains only short-period terms and will not be considered in this section.

Now, the function

$$\overline{R} = \sum_{\ell \geq 2} N_{☾}^{2} a_{☾}^{2-\ell} \overline{R}_{\ell} \tag{29}$$

does not depend on M and is an explicit function of time only through $r_{☾}$, $v_{☾}$, and $\Omega_{☾}$, considering $I_{☾}$ and ε constants, which is a good approximation.

The integration of the pertinent equations can be performed numerically by using as input lunar ecliptic coordinates – or, for that matter, equatorial coordinates – stored in tapes. This will produce precise evaluation of the true lunar motion. However, such a method can be very expensive in time. A good approximation can be obtained by considering $I_{☾}$, $e_{☾}$, $a_{☾}$, and ε fixed values and $M_{☾}$, $\omega_{☾}$, and $\Omega_{☾}$ linear functions of time, as given in Section 1, neglecting accelerations of these elements. Also, an expansion in power series of $e_{☾}$ will converge rapidly owing to the small value of this eccentricity.

Along these lines, we can consider the expansions

$$\left(\frac{a_{☾}}{r_{☾}}\right)^{\ell+1} \begin{bmatrix}\sin\\ \cos\end{bmatrix} (\theta'_{\ell s}) = \sum_{k=-\infty}^{\infty} G_{\ell qk}(e_{☾}) \begin{bmatrix}\sin\\ \cos\end{bmatrix} \left(\theta^{*}_{\ell qsk}\right) , \tag{30}$$

where the G's are Kaula's (1961) coefficients, which in turn can be written as Hansen's coefficients

$$G_{\ell qk} = X_{\ell-2q+k}^{-(\ell+1),\ \ell-2q} \tag{31}$$

and

$$\theta^{*}_{\ell qsk} = (\ell - 2q)\ \omega_{☾} + (\ell - 2q + k)\ M_{☾} + s\left(\Omega_{☾} + \frac{\pi}{2}\right) \quad . \tag{32}$$

We also remark that $G_{\ell qk} = O(e_{☾}^{|k|})$. These functions are given by

$$G_{\ell qk}(e_{☾}) = (1+\beta_{☾}^2)^{\ell} \sum_{j=-\infty}^{\infty} J_j\left[(\ell - 2q + k)\, e_{☾}\right] X_{\ell-2q+k,\, j}^{-\ell-1,\, \ell-2q}(\beta_{☾}) \quad , \tag{33}$$

where the $J_j(x)$ are Bessel functions with the usual definition

$$J_j(x) = \sum_{s=0}^{\infty} (-1)^s \frac{(x/2)^{j+2s}}{(j+s)!\; s!} \quad . \tag{34}$$

Also, for $k - j - m \geq 0$,

$$X_{k,j}^{\ell,m} = (-\beta_{☾})^{k-j-m} \binom{\ell+1-m}{k-j-m} F(k-j-\ell-1, -m-\ell-1, k-j-m+1; \beta_{☾}^2) \tag{35}$$

and, for $k - j - m \leq 0$,

$$X_{k,j}^{\ell,m} = (-\beta_{☾})^{-k+j+m} \binom{\ell+1+m}{-k+j+m} F(-k+j-\ell-1, m-\ell-1, -k+j+m+1; \beta_{☾}^2) \quad . \tag{36}$$

Here again, the hypergeometric series terminate; that is, they are polynomials. However, the G's are infinite series that converge for all $e_{☾} < 1$, although for large $e_{☾}$ the convergence is slow.

It follows that

$$\overline{R}_{\ell} = \sum_{m=0}^{\ell} \sum_{s=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=0}^{\ell} \sum_{k=-\infty}^{\infty} a^{\ell} \frac{(-1)^m \epsilon_m \epsilon_s (\ell - s)!}{(\ell + m)!} F_{\ell mp}(I) F_{\ell sq}(I_{☾}) \times$$

(Eq. cont. on next page)

$$\times (1+\beta^2)^{-\ell-1} H_{\ell p(2p-\ell)}(\beta)\, G_{\ell qk}(e_{☾}) \times$$

$$\times \left[(-1)^{\ell+m-s} U_\ell^{m,-s} \cos(\overline{\theta}_{\ell pm} + \theta^*_{\ell qsk}) + U_\ell^{m,s} \cos(\overline{\theta}_{\ell pm} - \theta^*_{\ell qsk})\right] \quad . \tag{37}$$

In the foregoing expressions, the U's are functions of $\mathcal{E}$ only and, therefore, supposed to be constant.

The following particular terms will be defined:

$$\overline{R}_{\ell mspqk} = (-1)^m \frac{\epsilon_m \epsilon_s (\ell - s)!}{(\ell + m)!} F_{\ell sq}(I_{☾})\, G_{\ell qk}(e_{☾}) \times$$

$$\times a^\ell (1+\beta^2)^{-\ell-1} F_{\ell mp}(I)\, H_{\ell p(2p-\ell)}(\beta) \times$$

$$\times \left[(-1)^{\ell+m-s} U_\ell^{m,-s} C^+_{\ell mspqk} + U_\ell^{m,s} C^-_{\ell mspqk}\right] =$$

$$= \Phi_{\ell mspqk}(a, e, I) \left[(-1)^{\ell+m-s} U_\ell^{m,-s} C^+_{\ell mspqk} + U_\ell^{m,s} C^-_{\ell mspqk}\right] \quad , \tag{38}$$

where the definition of $\Phi_{\ell mspqk}$ is obvious and

$$C^\pm_{\ell mspqk} = \cos(\overline{\theta}_{\ell pm} \pm \theta^*_{\ell qsk}) \quad . \tag{39}$$

The following quantities will also be useful:

$$S^\pm_{\ell mspqk} = \sin(\overline{\theta}_{\ell pm} \pm \theta^*_{\ell qsk}) \quad , \tag{40}$$

$$D^{\pm}_{\ell mspqk} = (\ell - 2p)\,\dot{\omega} + m\dot{\Omega} \pm \left[(\ell - 2q)\,\dot{\omega}_{☾} + (\ell - 2q + k)\,\dot{M}_{☾} + s\,\dot{\Omega}_{☾}\right] , \tag{41}$$

and

$$\int C^{\pm}_{\ell mspqk}\, dt = S^{\pm}_{\ell mspqk} / D^{\pm}_{\ell mspqk} . \tag{42}$$

In the above relations, we are using the secular rates of the Moon's motion as given in Section 1 and, for the satellite, as given by the even zonal-harmonics coefficients. The dominant terms follow:

$$\dot{\omega} = -\frac{3}{4} J_2 \left(\frac{a_e}{a}\right)^2 \frac{1 - 5\cos^2 I}{(1-e^2)^2} ,$$

$$\dot{\Omega} = -\frac{3}{2} J_2 \left(\frac{a_e}{a}\right)^2 \frac{\cos I}{(1-e^2)^2} ,$$

$$\dot{M} = n\left[1 + \frac{3}{4} J_2 \left(\frac{a_e}{a}\right)^2 \frac{-1 + 3\cos^2 I}{(1-e^2)^{3/2}}\right] . \tag{43}$$

The relation between a and n is the perturbed Kepler law

$$n^2 a^3 = Gm_{\oplus}\left[1 + \frac{3}{4} J_2 \left(\frac{a_e}{a}\right)^2 \frac{1 - 3\cos^2 I}{(1-e^2)^{3/2}}\right] . \tag{44}$$

Now, let $\delta_1 e_{\ell mspqk}, \ldots, \delta_1 \overline{\Omega}_{\ell mspqk}$ be the linear long-period and secular perturbations that are obtained from Equations (27), integrating the right-hand members (a, e, I fixed) and substituting $N^2_{☾}\, a^{2-\ell}_{☾}\, \overline{R}_{\ell mspqk}$ for R. The partial derivatives entering Lagrange's Equations (27) are given by

$$\frac{\partial \overline{R}_{\ell mspqk}}{\partial \omega} = \Phi_{\ell mspqk}(2p - \ell)\left[(-1)^{\ell + m - s}\, U^{m,-s}_{\ell}\, S^{+}_{\ell mspqk} + U^{m,s}_{\ell}\, S^{-}_{\ell mspqk}\right] , \tag{45}$$

$$\frac{\partial \overline{R}_{\ell mspqk}}{\partial \Omega} = -\Phi_{\ell mspqk}\, m\left[(-1)^{\ell+m-s}\, U_{\ell}^{m,-s}\, S^{+}_{\ell mspqk} + U_{\ell}^{m,s}\, S^{-}_{\ell mspqk}\right] , \tag{46}$$

$$\frac{\partial \overline{R}_{\ell mspqk}}{\partial a} = \frac{\ell}{a}\, \overline{R}_{\ell mspqk} , \tag{47}$$

$$\frac{\partial \overline{R}_{\ell mspqk}}{\partial I} = \overline{R}_{\ell mspqk}\, (F_{\ell mp} \rightarrow F^{I}_{\ell mp}) , \tag{48}$$

and

$$\frac{\partial \overline{R}_{\ell mspqk}}{\partial e} = \overline{R}_{\ell mspqk}\left[H_{\ell p(2p-\ell)} \rightarrow H^{e}_{\ell p(2p-\ell)}\right] , \tag{49}$$

where, if only terms with positive powers are considered,

$$F^{I}_{\ell mp}(I) = \sum_{i} \frac{(2\ell-2i)!\; 2^{2i-2\ell}}{i!\; (\ell-i)!\; (\ell-m-2i)!}\, \sin^{\ell-m-2i-1} I \times$$

$$\times \sum_{j} \binom{m}{j} \left[(\ell-m-2i)\cos^{2} I - j \sin^{2} I\right] \cos^{j-1} I \times$$

$$\times \sum_{k} \binom{\ell-m-2i+j}{k} \binom{m-j}{p-i-k} (-1)^{k-q} , \tag{50}$$

with the same summation conventions of Equation (9): for $2p-\ell > 0$,

$$H^{e}_{\ell p(2p-\ell)} = \frac{\beta}{e\sqrt{1-e^{2}}} \left\{\left[(2p-\ell)\frac{1}{\beta} - \frac{\ell+1}{1+\beta^{2}}\right] H_{\ell p(2p-\ell)} + \right.$$

$$\left. + 2(1+\beta^{2})(-\beta)^{2p-\ell+1} \binom{2p+1}{2p-\ell} \frac{(\ell+1)(2p-2\ell-1)}{2p-\ell}\, F(-\ell, 2p-2\ell, 2p-\ell+1; \beta^{2})\right\} \tag{51}$$

and, for $2p-\ell \le 0$,

$$H^{e}_{\ell p(2p-\ell)} = \frac{\beta}{e\sqrt{1-e^2}}\left\{\left[(\ell-2p)\frac{1}{\beta}-\frac{\ell+1}{1+\beta^2}\right] H_{\ell p(2p-\ell)} -\right.$$

$$\left. - 2(1+\beta^2)(-\beta)^{\ell-2p+1}\binom{2\ell-2p+1}{\ell-2p}\frac{(\ell+1)(2p+1)}{\ell-2p+2} F(-\ell,-2p,\ell-2p+2;\beta^2)\right\} . \tag{52}$$

The following definitions are introduced:

$$\Phi_{\ell mspqk} = (-1)^m \frac{\epsilon_m \epsilon_s (\ell-s)!}{(\ell+m)!} F_{\ell sq}(I_☾)\, G_{\ell qk}(e_☾) \times$$

$$\times\, a^{\ell} (1+\beta^2)^{-\ell-1} F_{\ell mp}(I)\, H_{\ell p(2p-\ell)}(\beta) \quad , \tag{53}$$

$$\Phi^{e}_{\ell mspqk} = \Phi_{\ell mspqk}(H \to H^{e}) \quad , \tag{54}$$

$$\Phi^{I}_{\ell mspqk} = \Phi_{\ell mspqk}(F \to F^{I}) \quad , \tag{55}$$

$$C_{\ell mspqk} = (-1)^{\ell+m-s}\, U_{\ell}^{m,-s}\, C^{+}_{\ell mspqk} + U_{\ell}^{m,s}\, C^{-}_{\ell mspqk} \quad , \tag{56}$$

$$C^{\omega}_{\ell mspqk} = (\ell-2p)\left[(-1)^{\ell+m-s}\, U_{\ell}^{m,-s}\, C^{+}_{\ell mspqk}/D^{+}_{\ell mspqk} + \right.$$

$$\left. + U_{\ell}^{m,s}\, C^{-}_{\ell mspqk}/D^{-}_{\ell mspqk}\right] \quad , \tag{57}$$

$$C^{\Omega}_{\ell mspqk} = m\left[(-1)^{\ell+m-s}\, U_{\ell}^{m,-s}\, C^{+}_{\ell mspqk}/D^{+}_{\ell mspqk} + \right.$$

$$\left. + U_{\ell}^{m,s}\, C^{-}_{\ell mspqk}/D^{-}_{\ell mspqk}\right] \quad ; \tag{58}$$

and

$$S_{\ell mspqk} = (-1)^{\ell+m-s}\, U_{\ell}^{m,-s}\, S^{+}_{\ell mspqk}/D^{+}_{\ell mspqk} + U_{\ell}^{m,s}\, S^{-}_{\ell mspqk}/D^{-}_{\ell mspqk} \quad . \tag{59}$$

Thus, we can write

$$\int \frac{\partial \overline{R}_{\ell mspqk}}{\partial \omega}\, dt = \Phi_{\ell mspqk}\, C^{\omega}_{\ell mspqk} \quad ,$$

$$\int \frac{\partial \overline{R}_{\ell mspqk}}{\partial \Omega}\, dt = \Phi_{\ell mspqk}\, C^{\Omega}_{\ell mspqk} \quad ,$$

$$\int \frac{\partial \overline{R}_{\ell mspqk}}{\partial a}\, dt = \frac{\ell}{a}\, \Phi_{\ell mspqk}\, S_{\ell mspqk} \quad ,$$

$$\int \frac{\partial \overline{R}_{\ell mspqk}}{\partial e}\, dt = \Phi^{e}_{\ell mspqk}\, S_{\ell mspqk} \quad ,$$

$$\int \frac{\partial \overline{R}_{\ell mspqk}}{\partial I}\, dt = \Phi^{I}_{\ell mspqk}\, S_{\ell mspqk} \quad .$$

The above integrals are not valid if the integers

$\ell - 2p, \ \ell - 2q + k, \ \ell - 2q, \ m, \ s$

are simultaneousiy zero; that is, we must exclude the cases

$$\begin{aligned} &m = 0 \quad , \\ &s = 0 \quad , \\ &2p = \ell = \text{even} = 2\gamma \quad , \\ &2q = \ell = \text{even} = 2\gamma \quad , \\ &k = 0 \quad . \end{aligned} \tag{60}$$

They correspond to secular perturbations and, in this case,

$$\frac{\partial \overline{R}_{2\gamma,0,0,\gamma,\gamma,0}}{\partial \omega} = \frac{\partial \overline{R}_{2\gamma,0,0,\gamma,\gamma,0}}{\partial \Omega} = 0 \quad .$$

In order to evaluate the other three integrals, we must consider (ℓ = even = 2γ):

$$F^{I}_{2\gamma,0,\gamma}(I) = \sum_{i=0}^{\gamma-1} \frac{(4\gamma - 2i)!\, 2^{2i-4\gamma}\, (2\gamma - 2i)}{i!\, (2\gamma - i)!\, (2\gamma - 2i)!} \sin^{2\gamma-2i-1} I \cos I \quad , \tag{61}$$

$$H_{2\gamma,\gamma,0} = F(-2\gamma - 1, -2\gamma - 1, 1; \beta^2) = \sum_{n=0}^{2\gamma+1} \frac{(-2\gamma - 1)_n\, (-2\gamma - 1)_n}{n!} \beta^{2n} \quad , \tag{62}$$

and

$$H^{e}_{2\gamma,\gamma,0} = \frac{2}{e\sqrt{1-e^2}} \sum_{n=1}^{2\gamma+1} \frac{(-2\gamma - 1)_n\, (-2\gamma - 1)_n}{(n-1)!} \beta^{2n} \quad . \tag{63}$$

Therefore,

$$\frac{\partial \overline{R}_{2\gamma,0,0,\gamma,\gamma,0}}{\partial a} = \frac{2\gamma}{a} \overline{R}_{2\gamma,0,0,\gamma,\gamma,0} \quad ,$$

$$\frac{\partial \overline{R}_{2\gamma,0,0,\gamma,\gamma,0}}{\partial I} = \overline{R}_{2\gamma,0,0,\gamma,\gamma,0} \left(F_{2\gamma,0,\gamma} \rightarrow F^{I}_{2\gamma,0,\gamma}\right) \quad ,$$

$$\frac{\partial \overline{R}_{2\gamma,0,0,\gamma,\gamma,0}}{\partial e} = \overline{R}_{2\gamma,0,0,\gamma,\gamma,0} \left(H_{2\gamma,\gamma,0} \rightarrow H^{e}_{2\gamma,\gamma,0}\right) \quad .$$

The long-period perturbations are given (excluding cases (60)) by

$$\delta_1 \bar{e}_{\ell mspqk} = N_{\text{☾}}^2\, a_{\text{☾}}^{2-\ell} \left(- \frac{\sqrt{1-e^2}}{na^2 e} \Phi_{\ell mspqk}\, C^{\omega}_{\ell mspqk} \right) ,$$

$$\delta_1 \bar{I}_{\ell mspqk} = N_{\text{☾}}^2\, a_{\text{☾}}^{2-\ell} \frac{1}{na^2 \sqrt{1-e^2}} \left(\cot I\, C^{\omega}_{\ell mspqk} - \operatorname{cosec} I\, C^{\Omega}_{\ell mspqk} \right) \times$$

$$\times \Phi_{\ell mspqk} ,$$

$$\delta_1 \bar{M}_{\ell mspqk} = N_{\text{☾}}^2\, a_{\text{☾}}^{2-\ell} \frac{1}{na^2} \left(- \frac{1-e^2}{e} \Phi^{e}_{\ell mspqk} - 2\ell\, \Phi_{\ell mspqk} \right) S_{\ell mspqk} ,$$

$$\delta_1 \bar{\omega}_{\ell mspqk} = N_{\text{☾}}^2\, a_{\text{☾}}^{2-\ell} \frac{1}{na^2} \left(- \frac{\cot I}{\sqrt{1-e^2}} \Phi^{I}_{\ell mspqk} + \frac{\sqrt{1+e^2}}{e} \Phi^{e}_{\ell mspqk} \right) S_{\ell mspqk} ,$$

$$\delta_1 \bar{\Omega}_{\ell mspqk} = N_{\text{☾}}^2\, a_{\text{☾}}^{2-\ell} \frac{\operatorname{cosec} I}{na^2 \sqrt{1-e^2}} K^{I}_{\ell mspqk}\, S_{\ell mspqk} . \qquad (64)$$

The secular perturbations are given by

$$\delta_1 \bar{M}_{2\gamma,0,0,\gamma,\gamma,0} = -N_{\text{☾}}^2\, a_{\text{☾}}^{2-2\gamma} \frac{1}{na^2} \left[\frac{1-e^2}{e} \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} (H \rightarrow H^{e}) + \right.$$

$$\left. + 4\gamma\, \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} \right] t ,$$

$$\delta_1 \bar{\omega}_{2\gamma,0,0,\gamma,\gamma,0} = N_{\text{☾}}^2\, a_{\text{☾}}^{2-2\gamma} \frac{1}{na^2} \left[- \frac{\cot I}{\sqrt{1-e^2}} \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} (F \rightarrow F^{I}) + \right.$$

$$\left. + \frac{\sqrt{1-e^2}}{e} \bar{R}_{2\gamma,0,0,\gamma,\gamma,0} (H \rightarrow H^{e}) \right] t ,$$

(Eq. cont. on next page)

$$\delta_1 \overline{\Omega}_{2\gamma,0,0,\gamma,\gamma,0} = N_{☾}^2 \, a_{☾}^{2-2\gamma} \frac{\operatorname{cosec} I}{na^2\sqrt{1-e^2}} \left[\overline{R}_{2\gamma,0,0,\gamma,\gamma,0} \, (F \rightarrow F^I) \right] t \quad . \tag{65}$$

6. SECULAR AND LONG-PERIOD SECOND-ORDER PERTURBATIONS

In a second-order evaluation, that is, if terms of the order $J_2 N_☾^2 a_☾^{2-\ell}$ are considered, it will be necessary to take into account the secular changes in M, ω, Ω owing to J_2. Such changes produce the largest higher order perturbations since they produce amplitudes that increase linearly with time. Let $\delta_2 \overline{M}$, $\delta_2 \overline{\omega}$, and $\delta_2 \overline{\Omega}$ be these perturbations. Taking into account only secular coefficients and noting that $\delta_1 \overline{a} = 0$, we have

$$\frac{d}{dt} (\delta_2 \overline{M}) = \frac{\partial \dot{M}}{\partial e} \delta_1 \overline{e} + \frac{\partial \dot{M}}{\partial I} \delta_1 \overline{I} \quad ,$$

$$\frac{d}{dt} (\delta_2 \overline{\omega}) = \frac{\partial \dot{\omega}}{\partial e} \delta_1 \overline{e} + \frac{\partial \dot{\omega}}{\partial I} \delta_1 \overline{I} \quad ,$$

$$\frac{d}{dt} (\delta_2 \overline{\Omega}) = \frac{\partial \dot{\Omega}}{\partial e} \delta_1 \overline{e} + \frac{\partial \dot{\Omega}}{\partial I} \delta_1 \overline{I} \quad .$$

By considering Equations (43), we find that

$$\frac{d}{dt} (\delta_2 \overline{M}) = 3e \sqrt{1 - e^2} \, \frac{1 - 3 \cos^2 I}{1 - 5 \cos^2 I} \, \dot{\omega} \, \delta_1 \overline{e} + 3 \sqrt{1 - e^2} \sin I \, \dot{\Omega} \, \delta_1 \overline{I} \quad ,$$

$$\frac{d}{dt} (\delta_2 \overline{\omega}) = \frac{4e}{1 - e^2} \, \dot{\omega} \, \delta_1 \overline{e} + 5 \dot{\Omega} \sin I \, \delta_1 \overline{I} \quad ,$$

$$\frac{d}{dt} (\delta_2 \overline{\Omega}) = \frac{4e}{1 - e^2} \, \dot{\Omega} \, \delta_1 \overline{e} - \dot{\Omega} \tan I \, \delta_1 \overline{I} \quad , \tag{66}$$

where, again, $\dot{\omega}$, $\dot{\Omega}$ are given by Equations (43). It follows that

$$\delta_2 \overline{M} = 3e\sqrt{1-e^2}\ \frac{1 - 3\cos^2 I}{1 - 5\cos^2 I}\ \dot{\omega} \int \delta_1 \overline{e}\, dt + 3\sqrt{1-e^2}\ \dot{\Omega} \sin I \int \delta_1 \overline{I}\, dt \quad ,$$

$$\delta_2 \overline{\omega} = \frac{4e}{1-e^2}\ \dot{\omega} \int \delta_1 \overline{e}\, dt + 5\dot{\Omega} \sin I \int \delta_1 \overline{I}\, dt \quad ,$$

$$\delta_2 \overline{\Omega} = \frac{4e}{1-e^2}\ \dot{\Omega} \int \delta_1 \overline{e}\, dt - \dot{\Omega} \tan I \int \delta_1 \overline{I}\, dt \quad . \tag{67}$$

Secular accelerations do not exist, since $\overline{e}$, $\overline{I}$ have only long-period terms. Therefore, conditions (60) have to be excluded. If we consider the first two Equations of (64), it follows that

$$\int C^{\omega}_{\ell mspqk}\, dt = S^{\omega}_{\ell mspqk} \quad ,$$

$$\int C^{\Omega}_{\ell mspqk}\, dt = S^{\Omega}_{\ell mspqk} \quad ,$$

where

$$S^{\omega}_{\ell mspqk} = \Big[(-1)^{\ell + m - s}\, U_{\ell}^{m, -s}\, S^{+}_{\ell mspqk} / (D^{+}_{\ell mspqk})^2 +$$

$$+ U_{\ell}^{m, s}\, S^{-}_{\ell mspqk} / (D^{-}_{\ell mspqk})^2 \Big] (\ell - 2p) \quad ,$$

$$S^{\Omega}_{\ell mspqk} = \Big[(-1)^{\ell + m - s}\, U_{\ell}^{m, -s}\, S^{+}_{\ell mspqk} / (D^{+}_{\ell mspqk})^2 +$$

$$+ U_{\ell}^{m, s}\, S^{-}_{\ell mspqk} / (D^{-}_{\ell mspqk})^2 \Big]\, m \quad . \tag{68}$$

It follows that

$$\delta_2 \overline{M}_{\ell mspqk} = N_{☾}^2 a_{☾}^{2-\ell} \frac{3\sqrt{1-e^2}}{na^2} \left(- \frac{1 - 3\cos^2 I}{1 - 5\cos^2 I} \sqrt{1-e^2}\,\dot{\omega} + \frac{\cos I}{\sqrt{1-e^2}}\,\dot{\Omega} \right) \times$$

$$\times\, \Phi_{\ell mspqk}\, S^{\omega}_{\ell mspqk} - N_{☾}^2 a_{☾}^{2-\ell} \frac{3}{na^2}\,\dot{\Omega}\, \Phi_{\ell mspqk}\, S^{\Omega}_{\ell mspqk} \quad ,$$

$$\delta_2 \overline{\omega}_{\ell mspqk} = N_{☾}^2 a_{☾}^{2-\ell} \frac{1}{na^2\sqrt{1-e^2}} (-4\,\dot{\omega} + 5\,\dot{\Omega}\cos I)\, \Phi_{\ell mspqk}\, S^{\omega}_{\ell mspqk} -$$

$$- N_{☾}^2 a_{☾}^{2-\ell} \frac{5\,\dot{\Omega}}{na^2\sqrt{1-e^2}}\, \Phi_{\ell mspqk}\, S^{\Omega}_{\ell mspqk} \quad ,$$

$$\delta_2 \overline{\Omega}_{\ell mspqk} = -N_{☾}^2 a_{☾}^{2-\ell} \frac{5\,\dot{\Omega}}{na^2\sqrt{1-e^2}}\, \Phi_{\ell mspqk}\, S^{\omega}_{\ell mspqk} +$$

$$+ N_{☾}^2 a_{☾}^{2-\ell} \frac{\dot{\Omega}\sec I}{na^2\sqrt{1-e^2}}\, \Phi_{\ell mspqk}\, S^{\Omega}_{\ell mspqk} \quad . \tag{69}$$

The total long-period and secular perturbations, including leading coupling terms with J_2, are finally obtained by

$$\delta\overline{e} = \delta_1\,\overline{e} + \delta_2\,\overline{e} \quad , \ldots , \quad \delta\overline{\Omega} = \delta_1\overline{\Omega} + \delta_2\overline{\Omega} \quad .$$

Obviously, the above relations are not valid for cases of critical inclination or satellites whose periods are commensurable with the rotation period (24 h) of the Earth.

7. COMPUTATIONAL PROCEDURE FOR LONG-PERIOD AND SECULAR PERTURBATIONS

In short, to obtain long-period and secular perturbations due to a term $\overline{R}_{\ell mspqk}$ (Eq. 38), we proceed as follows:

7.1 Long-Period Perturbations

Compute, given means elements a, e, I, $e_☾$, $I_☾$, $\mathcal{E}$, and J_2:

1) $N_☾^2 \, a_☾^{2-\ell}$

2) β (22)

3) $F_{\ell sq}(I_☾)$ (9)

4) $F_{\ell mp}(I)$ (9)

5) $G_{\ell qk}(e_☾)$ (33), (34), (35), or (36)

6) $H_{\ell p(2p-\ell)}(\beta)$ (23) or (24)

7) $\Phi_{\ell mspqk}$ (53)

8) $U_\ell^{m,-s}$, $U_\ell^{m,s}$ (12a) or (12b) ($\mathcal{E}$ from Section 1)

9) ω, Ω (43)

10) $\overline{\theta}_{\ell pm}$ (26)

11) $\omega_☾$, $M_☾$, $\Omega_☾$ (Section 1)

12) $\theta^*_{\ell qsk}$ (32)

13) $C^{\pm}_{\ell mspqk}$ (39)

14) $S^{\pm}_{\ell mspqk}$ (40)

15) $\dot{\omega}_{☾}$, $\dot{\Omega}_{☾}$, $\dot{M}_{☾}$ (Section 1)

16) $\dot{\omega}$, $\dot{\Omega}$, $\dot{M}$ (43)

17) $D^{\pm}_{\ell mspqk}$ (41)

18) $F^{I}_{\ell mp}$ (51)

19) $H^{e}_{\ell p(2p-\ell)}$ (51) or (52)

20) $\Phi^{e}_{\ell mspqk}$ (54)

21) $\Phi^{I}_{\ell mspqk}$ (55)

22) $C_{\ell mspqk}$ (56)

23) $S_{\ell mspqk}$ (59)

24) $C^{\omega}_{\ell mspqk}$ (57)

25) $C^{\Omega}_{\ell mspqk}$ (58)

26) $S^{\omega}_{\ell mspqk}$ (68)

27) $S^{\Omega}_{\ell mspqk}$ (68)

28) δ_1 (element)$_{\ell mspqk}$ (64)

29) δ_2 (element)$_{\ell mspqk}$ (69)

30) δ (element) = δ_1 (element) + δ_2 (element)
Complete long-period perturbations.

7.2 Secular Perturbations (ℓ even = 2γ)

Given a, e, I, $e_{☾}$, $I_{☾}$, $\mathcal{E}$, β (mean values):

1) $F_{2\gamma, 0, \gamma}(I_{☾})$ (9)

2) $F_{2\gamma, 0, \gamma}(I)$ (9)

3) $G_{2\gamma,0,0}(e_☾)$ (33), (34), (35), or (36)

4) $H_{2\gamma,\gamma,0}(\beta)$ (62)

5) $U_{2\gamma}^{0,0}$ (12b)

6) $\Phi_{2\gamma,0,0,\gamma,\gamma,0}$ (53)

7) $C^{\pm}_{2\gamma,0,0,\gamma,\gamma,0} = 1$

8) $\overline{R}_{2\gamma,0,0,\gamma,\gamma,0}$ (38)

9) $F^{I}_{2\gamma,0,\gamma}(I)$ (61)

10) $H^{e}_{2\gamma,\gamma,0}(\beta)$ (63)

11) $\overline{R}_{2\gamma,0,0,\gamma,\gamma,0}(F \to F^{I})$ (38)

12) $\overline{R}_{2\gamma,0,0,\gamma,\gamma,0}(H \to H^{e})$ (38)

13) δ_1 (element, secular) (65)
Complete secular perturbations.

8. SHORT-PERIOD PERTURBATIONS OF LOW SATELLITES

During a few revolutions of the satellite, where short-period variations are of interest, the position of the Moon and ω, Ω change little for low satellites. Then, as Kozai (1966) suggested, we can consider the elements of the Moon, ω and Ω, fixed when performing the integrations. In this case, the appropriate expression for R_ℓ is given in Equation (8), since here it is immaterial what frame of reference is being used for the coordinates of the Moon. The particular term R_ℓ will be written as

$$R_\ell = a^\ell \left(\frac{r}{a}\right)^\ell \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} F_{\ell mp}(I) \{C_\ell^m \cos [(\ell - 2p) v + m\Omega] +$$

$$+ S_\ell^m \sin [(\ell - 2p) v + m\Omega]\} \quad , \tag{70}$$

where

$$C_\ell^m = \begin{cases} A_\ell^m \,, & \ell - m \text{ even} \\ -B_\ell^m \,, & \ell - m \text{ odd} \end{cases} \,,$$

$$S_\ell^m = \begin{cases} B_\ell^m \,, & \ell - m \text{ even} \\ A_\ell^m \,, & \ell - m \text{ odd} \end{cases} \,. \tag{71}$$

The coefficients C's and S's depend only on the Moon (5). The values of $r_☾$, v', and Ω', given mean elements $a_☾$, $e_☾$, $I_☾$, ε, and the time T, can be computed by considering $\omega_☾$, $M_☾$, and $\Omega_☾$ (Section 1), then solving Kepler's equation

$$M_☾ = E_☾ - e_☾ \sin E_☾ \quad , \tag{72}$$

computing $f_☾$ from

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$$\tan \frac{f_☾}{2} = \sqrt{\frac{1+e_☾}{1-e_☾}} \tan \frac{E_☾}{2} \quad , \tag{73}$$

computing

$$r_☾ = a_☾ (1 - e_☾ \cos E_☾)$$

$$v_☾ = f_☾ + \omega_☾ \quad ,$$

computing $\alpha_☾$, $\delta_☾$ from Equations (7), and finally α', δ' from Equations (10).

Let

$$R_{\ell mp} = a^\ell \left(\frac{r}{a}\right)^\ell F_{\ell mp}(I) \left(C_\ell^m \cos \theta_{\ell pm} + S_\ell^m \sin \theta_{\ell pm}\right) \quad , \tag{74}$$

where

$$\theta_{\ell pm} = (\ell - 2p) v + m\Omega \quad .$$

The following relations are easily established:

$$\frac{\partial R_{\ell mp}}{\partial a} = \frac{\ell}{a} R_{\ell mp} \quad ; \tag{75}$$

$$\frac{\partial R_{\ell mp}}{\partial e} = \ell \frac{e^2 - 1}{e} \left(\frac{a}{r}\right)^2 R_{\ell mp} + \ell \frac{1}{e} \left(\frac{a}{r}\right) R_{\ell mp} +$$

$$+ (\ell - 2p) a^\ell \left(\frac{r}{a}\right)^{\ell - 1} F_{\ell mp}(I) \left(-C_\ell^m \sin \theta_{\ell pm} +\right.$$

$$\left. + S_\ell^m \cos \theta_{\ell pm}\right) \sin f \left(1 + \frac{r}{a} \frac{1}{1 - e^2}\right) \quad ; \tag{76}$$

$$\frac{\partial R_{\ell mp}}{\partial I} = a^{\ell} \left(\frac{r}{a}\right)^{\ell} F^{I}_{\ell mp}(I) \left(C_{\ell}^{m} \cos \theta_{\ell pm} + S_{\ell}^{m} \sin \theta_{\ell pm}\right) \quad , \tag{77}$$

where $F^{I}_{\ell mp}(I)$ is given by Equation (50);

$$\frac{\partial R_{\ell mp}}{\partial \omega} = (\ell - 2p)\, a^{\ell} \left(\frac{r}{a}\right)^{\ell} F_{\ell mp}(I) \left(-C_{\ell}^{m} \sin \theta_{\ell pm} + S_{\ell}^{m} \cos \theta_{\ell pm}\right) \quad ; \tag{78}$$

$$\frac{\partial R_{\ell mp}}{\partial \Omega} = m\, a^{\ell} \left(\frac{r}{a}\right)^{\ell} F_{\ell mp}(I) \left(-C_{\ell}^{m} \sin \theta_{\ell pm} + S_{\ell}^{m} \cos \theta_{\ell pm}\right) \quad ; \tag{79}$$

and

$$\frac{\partial R_{\ell mp}}{\partial M} = \ell \frac{e}{\sqrt{1-e^{2}}} \frac{a}{r} R_{\ell mp} \sin f +$$

$$+ (\ell - 2p)\, a^{\ell} \sqrt{1-e^{2}} \left(\frac{r}{a}\right)^{\ell - 2} \left(-C_{\ell}^{m} \sin \theta_{\ell pm} + S_{\ell}^{m} \cos \theta_{\ell pm}\right) F_{\ell mp}(I) \quad . \tag{80}$$

For short-period perturbations, we make use of Lagrange's equations (27) where R is substituted by

$$R_{per.} = R - \frac{1}{2\pi} \int_{0}^{2\pi} R \, dM \quad , \tag{81}$$

and integration is carried on with respect to $dt = dM/n$, considering all other angles and actions to be constant.

The computation of the average of R with respect to M involves the following integrals:

$$I_1^{\ell,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\ell} \cos(\ell-2p)f\,dM = (1+\beta^2)^{-\ell-1} X_{0,0}^{\ell,\ell-2p}(\beta) \quad , \tag{82}$$

$$I_2^{\ell,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\ell-2} \cos(\ell-2p)f\,dM = (1+\beta^2)^{-\ell+1} X_{0,0}^{\ell-2,\ell-2p}(\beta) \quad , \tag{83}$$

$$I_3^{\ell,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\ell-1} \cos(\ell-2p)f\,dM = (1+\beta^2)^{-\ell} X_{0,0}^{\ell-1,\ell-2p}(\beta) \quad ; \tag{84}$$

$$I_4^{\ell,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\ell-1} \sin(\ell-2p)f\,\sin f\,dM =$$

$$= (1+\beta^2)^{-\ell}\left(\frac{1}{2}X_{0,0}^{\ell-1,\ell-2p-1} - \frac{1}{2}X_{0,0}^{\ell-1,\ell-2p+1}\right) \quad , \tag{85}$$

and

$$I_5^{\ell,p} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^{\ell} \sin(\ell-2p)f\,\sin f\,dM =$$

$$= (1+\beta^2)^{-\ell-1}\left(\frac{1}{2}X_{0,0}^{\ell,\ell-2p-1} - \frac{1}{2}X_{0,0}^{\ell,\ell-2p+1}\right) \quad , \tag{86}$$

where $X_{0,0}^{k,j}(\beta)$ is Hansen's coefficient defined by Equation (23) or (24).

It follows that, by defining

$$\overline{S}_{\ell pm} = -C_\ell^m \sin\overline{\theta}_{\ell pm} + S_\ell^m \cos\overline{\theta}_{\ell pm} \quad , \tag{87}$$

and

$$\overline{C}_{\ell pm} = C_\ell^m \cos\overline{\theta}_{\ell pm} + S_\ell^m \sin\overline{\theta}_{\ell pm} \quad , \tag{88}$$

where $\overline{\theta}_{\ell pm}$ is given by Equation (26), we have

$$I^{(1)}_{\ell mp} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{\ell mp}}{\partial M} \, dM = 0 \quad ,$$

$$I^{(2)}_{\ell mp} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{\ell mp}}{\partial \omega} \, dM = (\ell - 2p) \, a^{\ell} \, F_{\ell mp}(I) \, I_1^{\ell, p} \, \overline{S}_{\ell pm} \quad , \tag{89}$$

$$I^{(3)}_{\ell mp} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{\ell mp}}{\partial \Omega} \, dM = m \, a^{\ell} \, F_{\ell mp}(I) \, I_1^{\ell, p} \, \overline{S}_{\ell pm} \quad , \tag{90}$$

$$I^{(4)}_{\ell mp} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{\ell mp}}{\partial e} \, dM = \ell \, \frac{e^2 - 1}{e} \, a^{\ell} \, F_{\ell mp}(I) \, I_2^{\ell, p} \, \overline{C}_{\ell pm} +$$

$$+ \ell \, \frac{1}{e} \, a^{\ell} \, F_{\ell mp}(I) \, I_3^{\ell, p} \, \overline{C}_{\ell mp} - (\ell - 2p) \, a^{\ell} \, F_{\ell mp}(I) \, I_4^{\ell, p} \, \overline{C}_{\ell pm} -$$

$$- (\ell - 2p) \, a^{\ell} \, \frac{1}{1 - e^2} \, F_{\ell mp}(I) \, I_5^{\ell, p} \, \overline{C}_{\ell pm} \quad , \tag{91}$$

$$I^{(5)}_{\ell mp} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{\ell mp}}{\partial a} \, dM = 2\ell \, a^{\ell - 1} \, F_{\ell mp}(I) \, I_1^{\ell, p} \, \overline{C}_{\ell pm} \quad , \tag{92}$$

and

$$I^{(6)}_{\ell mp} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial R_{\ell mp}}{\partial I} \, dM = a^{\ell} \, F^{I}_{\ell mp}(I) \, I_1^{\ell, p} \, \overline{C}_{\ell pm} \quad . \tag{93}$$

Now it is necessary to evaluate, in closed form, integrals of the type

$$J^R_{pq} + i \, J^I_{pq} \equiv J_{pq} = \int \left(\frac{r}{a}\right)^p e^{iqf} \, dM \tag{94}$$

for $q = 0, 1, 2, \ldots, p+1$ and $p = 0, 1, 2, \ldots$. Introducing $dM = (r/a) \, dE$, we obtain

$$J_{pq} = \int \left(\frac{r}{a}\right)^{p+1} e^{iqf} \, dE \quad ,$$

where

$$\left(\frac{r}{a}\right)^{p+1} = (1 - e \cos E)^{p+1} \quad ,$$

$$e^{iqf} = \left(\frac{a}{r}\right)^{q} [(\cos E - e) + i\eta \sin E]^{q} \quad ,$$

$$\eta = \sqrt{1 - e^2} \quad , \qquad i = \sqrt{-1} \quad .$$

Therefore,

$$J_{pq} = \int \left(\frac{r}{a}\right)^{p+1-q} [(\cos E - e) + i\eta \sin E]^{q} \, dE$$

with

$$p + 1 - q \geq 0 \quad .$$

It is easily found that

$$J_{pq} = \sum_{\substack{\gamma=-p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma} K_{pq\gamma}(e) \, (\sin \gamma E - i \cos \gamma E) + K_{pq0}(e) \, E \quad , \tag{95}$$

where

$$K_{pq\gamma} = \sum_{\alpha} \sum_{k} \sum_{s} 2^{-\alpha} (-1)^{q-\alpha} \binom{p+1-q}{k} \binom{k}{\frac{\alpha-\gamma}{2} - s} \binom{q}{\alpha - k} \binom{\alpha - k}{s} \times$$

$$\times \, e^{2k+q-\alpha} (1+\eta)^{\alpha-k-s} (1-\eta)^{s} \quad , \tag{96}$$

in which

$$\alpha = |\gamma|, \ |\gamma|+2, \ |\gamma|+4, \ \ldots, \ p \text{ or } p+1 \quad ,$$

$$k = 0, 1, 2, \ldots, \frac{\alpha - \gamma}{2} \quad ,$$

$$s = 0, 1, 2, \ldots, \alpha - k \quad .$$

Thus,

$$J^R_{pq} \equiv \int \left(\frac{r}{a}\right)^p \cos qf \, dM = \sum_{\substack{\gamma=-p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma} K_{pq\gamma} \sin \gamma E + K_{pq0} E \tag{97}$$

and

$$J^I_{pq} = \int \left(\frac{r}{a}\right)^p \sin qf \, dM = - \sum_{\substack{\gamma=-p-1 \\ (\gamma \neq 0)}}^{p+1} \frac{1}{\gamma} K_{pq\gamma} \cos \gamma E \quad . \tag{98}$$

The following integrals are then established:

$$\begin{aligned} A_1^{\ell mp} \equiv \int \frac{\partial R_{\ell mp}}{\partial M} dM = \frac{\ell}{2} a^\ell \frac{e}{\sqrt{1-e^2}} F_{\ell mp}(I) \Big[\left(J^I_{\ell-1, \ell-2p-1} - J^I_{\ell-1, \ell-2p+1} \right) \overline{C}_{\ell pm} + \\ + \left(J^R_{\ell-1, \ell-2p-1} - J^R_{\ell-1, \ell-2p+1} \right) \overline{S}_{\ell pm} \Big] + (\ell - 2p) a^\ell \sqrt{1-e^2} F_{\ell mp}(I) \times \\ \times \left(-J^I_{\ell-2, \ell-2p} \overline{C}_{\ell pm} + J^R_{\ell-2, \ell-2p} \overline{S}_{\ell pm} \right) \quad , \end{aligned} \tag{99}$$

$$A_2^{\ell mp} \equiv \int \frac{\partial R_{\ell mp}}{\partial \omega} dM = (\ell - 2p) a^\ell F_{\ell mp}(I) \left(-J^I_{\ell, \ell-2p} \overline{C}_{\ell pm} + J^R_{\ell, \ell-2p} \overline{S}_{\ell pm} \right) \quad , \tag{100}$$

$$A_3^{\ell mp} \equiv \int \frac{\partial R_{\ell mp}}{\partial \Omega}\, dM = m\, a^\ell\, F_{\ell mp}(I) \left(-J^I_{\ell,\ell-2p}\, \overline{C}_{\ell pm} + J^R_{\ell,\ell-2p}\, \overline{S}_{\ell pm}\right) ,$$

(101)

$$A_4^{\ell mp} \equiv \int \frac{\partial R_{\ell mp}}{\partial e}\, dM = \ell\, a^\ell\, \frac{e^2-1}{e}\, F_{\ell mp}(I) \left(J^R_{\ell-2,\ell-2p}\, \overline{C}_{\ell pm} + J^I_{\ell-2,\ell-2p}\, \overline{S}_{\ell pm}\right) +$$

$$+ \ell\, \frac{1}{e}\, a^\ell\, F_{\ell mp}(I) \left(J^R_{\ell-1,\ell-2p}\, \overline{C}_{\ell pm} + J^I_{\ell-1,\ell-2p}\, \overline{S}_{\ell pm}\right) +$$

$$+ \frac{1}{2}\, a^\ell\, F_{\ell mp}(I) \Big[-\left(J^R_{\ell-1,\ell-2p-1} - J^R_{\ell-1,\ell-2p+1}\right) \overline{C}_{\ell pm} +$$

$$+ \left(J^I_{\ell-1,\ell-2p-1} - J^I_{\ell-1,\ell-2p+1}\right) \overline{S}_{\ell pm}\Big] +$$

$$+ \frac{1}{2}\, a^\ell\, \frac{1}{1-e^2}\, (\ell-2p)\, F_{\ell mp}(I) \Big[-\left(J^R_{\ell,\ell-2p-1} - J^R_{\ell,\ell-2p+1}\right) \overline{C}_{\ell pm} +$$

$$+ \left(J^I_{\ell,\ell-2p-1} - J^I_{\ell,\ell-2p+1}\right) \overline{S}_{\ell pm}\Big] ,$$

(102)

$$A_5^{\ell mp} \equiv \int \frac{\partial R_{\ell mp}}{\partial a}\, dM = \ell\, a^{\ell-1}\, F_{\ell mp}(I) \left(J^R_{\ell,\ell-2p}\, \overline{C}_{\ell pm} + J^I_{\ell,\ell-2p}\, \overline{S}_{\ell pm}\right) ,$$

(103)

and

$$A_6^{\ell mp} \equiv \int \frac{\partial R_{\ell mp}}{\partial I}\, dM = a^\ell\, F^I_{\ell mp}(I) \left(J^R_{\ell,\ell-2p}\, \overline{C}_{\ell pm} + J^I_{\ell,\ell-2p}\, \overline{S}_{\ell pm}\right) . \quad (104)$$

Now, if we make use of Lagrange's equations and consider Equation (81), the short-period perturbations due to a term $R_{\ell mp}$ are given by

$$\Delta a_{\ell mp} = N_{☾}^2\, a_{☾}^{2-\ell}\, \frac{2}{n^2 a}\, A_1^{\ell mp} ,$$

(Eq. cont. on next page)

$$\Delta e_{\ell mp} = \frac{N_{☾}^2 a_{☾}^{2-\ell}}{n^2 a^2 e} \sqrt{1-e^2} \left[\sqrt{1-e^2}\, A_1^{\ell mp} - \left(A_2^{\ell mp} - I_{\ell mp}^{(2)} M\right)\right] ,$$

$$\Delta I_{\ell mp} = \frac{N_{☾}^2 a_{☾}^{2-\ell}}{n^2 a^2 \sqrt{1-e^2}} \operatorname{cosec} I \left[\left(A_2^{\ell mp} - I_{\ell mp}^{(2)} M\right) \cos I - \left(A_3^{\ell mp} - I_{\ell mp}^{(3)} M\right)\right] ,$$

$$\Delta M_{\ell mp} = - \frac{N_{☾}^2 a_{☾}^{2-\ell}}{n^2 a^2 e} \left[(1-e^2) \left(A_4^{\ell mp} - I_{\ell mp}^{(4)} M\right) + 2ae \left(A_5^{\ell mp} - I_{\ell mp}^{(5)} M\right)\right] + \Delta' M_{\ell mp} ,$$

$$\Delta\omega_{\ell mp} = \frac{N_{☾}^2 a_{☾}^{2-\ell}}{n^2 a^2 e \sqrt{1-e^2}} \left[- e \cot I \left(A_6^{\ell mp} - I_{\ell mp}^{(6)} M\right) + (1-e^2) \left(A_4^{\ell mp} - I_{\ell mp}^{(4)} M\right)\right] ,$$

$$\Delta\Omega_{\ell mp} = \frac{N_{☾}^2 a_{☾}^{2-\ell} \operatorname{cosec} I}{n^2 a^2 \sqrt{1-e^2}} \left(A_6^{\ell mp} - I_{\ell mp}^{(6)} M\right) , \qquad (105)$$

where

$$\Delta' M_{\ell mp} = - \frac{3N_{☾}^2 a_{☾}^{2-\ell}}{n^2 a^2} \int A_1^{\ell mp}\, dM \qquad (106)$$

remains to be evaluated. This last involves the evaluation of the integral

$$L_{pq} = \int J_{pq}\, dM = \int J_{pq} (1 - e \cos E)\, dE = L_{pq}^R + i\, L_{pq}^I .$$

It is readily found that

$$L_{pq}^R = \int J_{pq}^R\, dM = - \sum_{\substack{\gamma=-p-1 \\ (\gamma\neq 0)}}^{p+1} \frac{1}{\gamma^2} K_{pq\gamma} \cos \gamma E + K_{pq0} \frac{E^2}{2} -$$

(Eq. cont. on next page)

$$- e K_{pq0} (E \sin E + \cos E) + \frac{e}{2} \sum_{\substack{\gamma=-p-1 \\ (\gamma\neq 0,\, \gamma\neq -1)}}^{p+1} \frac{1}{\gamma(\gamma+1)} K_{pq\gamma} \cos(\gamma+1) E +$$

$$+ \frac{e}{2} \sum_{\substack{\gamma=-p-1 \\ (\gamma\neq 0,\, \gamma\neq 1)}}^{p+1} \frac{1}{\gamma(\gamma-1)} K_{pq\gamma} \cos(\gamma-1) E \quad , \tag{107}$$

and

$$L^I_{pq} = \int J^I_{pq}\, dM = - \sum_{\substack{\gamma=-p-1 \\ (\gamma\neq 0)}}^{p+1} \frac{1}{\gamma^2} K_{pq\gamma} \sin\gamma E + \frac{e}{2} (K_{pq1} - K_{p,q,-1}) E +$$

$$+ \frac{e}{2} \sum_{\substack{\gamma=-p-1 \\ (\gamma\neq 0,\, \gamma\neq -1)}}^{p+1} \frac{1}{\gamma(\gamma+1)} K_{pq\gamma} \sin(\gamma+1) E + \frac{e}{2} \sum_{\substack{\gamma=-p-1 \\ (\gamma\neq 0,\, \gamma\neq 1)}}^{p+1} \frac{1}{\gamma(\gamma-1)} K_{pq\gamma} \sin(\gamma-1) E \quad . \tag{108}$$

Therefore,

$$\Delta' M_{\ell mp} = - \frac{3}{2} N^2_{☾} a^{2-\ell}_{☾} \frac{\ell a^{\ell-2}}{n^2} \frac{e}{\sqrt{1-e^2}} F_{\ell mp}(I) \Big[\big(L^I_{\ell-1,\ell-2p-1} -$$

$$- L^I_{\ell-1,\ell-2p+1} \big) \overline{C}_{\ell pm} + \big(L^R_{\ell-1,\ell-2p-1} - L^R_{\ell-1,\ell-2p+1} \big) \overline{S}_{\ell pm} \Big] -$$

$$- 3 N^2_{☾} a^{2-\ell}_{☾} (\ell - 2p) \frac{a^{\ell-2}}{n^2} \sqrt{1-e^2}\; F_{\ell mp}(I) \big(L^R_{\ell-2,\ell-2p} \overline{S}_{\ell pm} -$$

$$- L^I_{\ell-2,\ell-2p} \overline{C}_{\ell pm} \big) \quad . \tag{109}$$

9. COMPUTATIONAL PROCEDURE FOR SHORT-PERIOD PERTURBATIONS OF LOW SATELLITES

Corresponding to a particular term $R_{\ell mp}$ (Eq. 74), the short-period perturbations ($☾$, ω, Ω fixed) are computed as follows.

Given mean elements a, e, I, $e_☾$, $I_☾$, and ε, we compute

1) $N_☾^2 \, a_☾^{2-\ell}$

2) β (22)

3) Given $M_☾$, compute $E_☾$, $f_☾$

4) Given $\omega_☾$, $\Omega_☾$, and $a_☾$, compute $r_☾$, $v_☾$

5) $\alpha_☾$, $\delta_☾$ (7)

6) α', δ' (10)

7) $P_\ell^m (\sin \delta') \begin{matrix} \cos \\ \sin \end{matrix} m \delta'$

8) A_ℓ^m, B_ℓ^m (5)

9) C_ℓ^m, S_ℓ^m (71)

10) Given M, ω, and Ω, compute E

11) $\overline{\theta}_{\ell pm}$ (26)

12) $\overline{C}_{\ell pm}$, $\overline{S}_{\ell pm}$ (87), (88)

13) $F_{\ell mp}(I)$ (9)

14) $F^I_{\ell mp}(I)$ (50)

15) $X_{0,0}^{k,j}(\beta)$ (23) or (24)

k	j
ℓ	$\ell - 2p$
$\ell - 2$	$\ell - 2p$
$\ell - 1$	$\ell - 2p$
$\ell - 1$	$\ell - 2p - 1$
$\ell - 1$	$\ell - 2p + 1$
ℓ	$\ell - 2p - 1$
ℓ	$\ell - 2p + 1$

16) $K_{pq\gamma}(e)$ (96), $\gamma = -p-1, -p, \ldots, p, p+1$

p	q
$\ell - 1$	$\ell - 2p - 1$
$\ell - 1$	$\ell - 2p + 1$
$\ell - 2$	$\ell - 2p$
ℓ	$\ell - 2p$
$\ell - 1$	$\ell - 2p$
ℓ	$\ell - 2p - 1$
ℓ	$\ell - 2p + 1$

17) $I_1^{\ell,p}, I_2^{\ell,p}, \ldots, I_5^{\ell,p}$ (82) through (86)

18) $I^{(2)}_{\ell mp}, I^{(3)}_{\ell mp}, \ldots, I^{(6)}_{\ell mp}$ (89) through (93)

19) J^R_{pq}, J^I_{pq} (97), (98) (same range for p, q as in (15))

20) $A_1^{\ell mp}, A_2^{\ell mp}, \ldots, A_6^{\ell mp}$ (99) through (104)

21) L^R_{pq}, L^I_{pq} (107), (108)

p	q
$\ell - 1$	$\ell - 2p - 1$
$\ell - 1$	$\ell - 2p + 1$
$\ell - 2$	$\ell - 2p$

22) $\Delta' M_{\ell mp}$ (109)

23) $\Delta e_{\ell mp}, \ldots, \Delta\Omega_{\ell mp}$ (105)
Short-period perturbations completed.

10. SHORT-PERIOD PERTURBATIONS OF HIGH SATELLITES WITH SMALL ECCENTRICITY

When the satellite is high, for example, close to a 24-h period, the Moon can no longer be considered fixed during a few revolutions of the satellite. Here we consider also the variations of ω, Ω, in contrast to what we have done in Section 8. In this case, the integrals found in that section have to take these variations into account. This can be done only if the eccentricity of the satellite is small so that power series in e will converge rapidly. Thus, the disturbing function obtained in Equation (19) has to be developed in terms of M, $M_☾$.

The following expansions are well known:

$$\left(\frac{a_☾}{r_☾}\right)^{\ell+1} \begin{matrix}\sin\\ \cos\end{matrix} \left[(\ell-2q)\,v_☾ + s\left(\Omega_☾ + \frac{\pi}{2}\right)\right] = \sum_{k=-\infty}^{\infty} G_{\ell qk}(e_☾) \begin{matrix}\sin\\ \cos\end{matrix}\, \theta^☾_{\ell sqk} \quad , \tag{110}$$

and

$$\left(\frac{r}{a}\right)^{\ell} \begin{matrix}\sin\\ \cos\end{matrix} \left[(\ell-2p)\,v + m\Omega\right] = \sum_{j=-\infty}^{\infty} H_{\ell pj}(e) \begin{matrix}\sin\\ \cos\end{matrix}\, \theta_{\ell mpj} \quad , \tag{111}$$

where

$$\theta^☾_{\ell sqk} = (\ell-2q)\,\omega_☾ + (\ell-2q+k)\,M_☾ + s\left(\Omega_☾ + \frac{\pi}{2}\right) \quad , \tag{112}$$

$$\theta_{\ell mpj} = (\ell-2p)\omega + (\ell-2p+j)M + m\Omega \quad , \tag{113}$$

and $H_{\ell pj}(e)$ are Kaula's (1962) coefficients. These can also be written in terms of Hansen's coefficients by

$$H_{\ell pj}(e) = X^{\ell,\,\ell-2p}_{\ell-2p+j}(\beta) \quad .$$

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The coefficients $G_{\ell qk}(e_☾)$ have been defined in Equations (33), (34), and (35) or (36).

The classical expressions for Kaula's coefficients are (e.g, Plummer, 1960, p. 45):

$$H_{\ell pj}(e) = (1+\beta^2)^{-\ell-1} \sum_{i=-\infty}^{\infty} J_i\,[(\ell-2p+j)e]\, X^{\ell,\ell-2p}_{\ell-2p+j,i} = O(e^{|j|}) \quad , \tag{114}$$

where $J_i(x)$ are the usual Bessel functions and $X^{\ell,m}_{k,i}(\beta)$ are given in terms of hypergeometric series (which always terminate), as follows: for $k-i-m \geq 0$,

$$X^{\ell,m}_{k,i} = (-\beta)^{k-i-m} \binom{\ell+1-m}{k-i-m} F(k-i-\ell-1, m-\ell-1, k-i-m+1; \beta^2) \tag{115}$$

and, for $k-i-m \leq 0$,

$$X^{\ell,m}_{k,i} = (-\beta)^{-k+i+m} \binom{\ell+1+m}{-k+i+m} F(-k+i-\ell-1, m-\ell-1, -k+i+m+1; \beta^2) \quad . \tag{116}$$

It follows that

$$R_\ell = \sum_{m=0}^{\ell} \sum_{s=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=0}^{\ell} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} R_{\ell mspqkj} \quad ,$$

where

$$R_{\ell mspqkj} = \frac{(-1)^m \epsilon_m \epsilon_s (\ell-s)!}{(\ell+m)!} a^\ell F_{\ell mp}(I)\, F_{\ell sq}(I_☾) \times$$

(Eq. cont. on next page)

$$\times\, G_{\ell qk}(e_{☾})\, H_{\ell pj}(e) \left[(-1)^{\ell+m-s}\, U_{\ell}^{m,-s} \cos(\theta_{\ell mpj} + \theta^{☾}_{\ell sqk}) + \right.$$

$$\left. + U_{\ell}^{m,s} \cos(\theta_{\ell mpj} - \theta^{☾}_{\ell sqk}) \right] \quad . \tag{117}$$

In what follows, it will be clear that we should have

$$\ell - 2n + j \neq 0 \quad .$$

We can make use of the fact that

$$\frac{\partial F(a, b, c; x)}{\partial x} = \frac{ab}{c} F(a+1,\ b+1,\ c+1; x)$$

and

$$\frac{\partial J_i(x)}{\partial x} = \frac{1}{2} [J_{i-1}(x) - J_{i+1}(x)] \quad ,$$

and arrive at the following relations:

$$Y_k^{\ell,m} \equiv \frac{\partial X_k^{\ell,m}}{\partial \beta} = \sum_{i<k-m} \left\{ J_i(ke)\, Y_{k,i}^{\ell,m} + \right.$$

$$\left. + \frac{e\sqrt{1-e^2}}{\beta} \frac{k}{2} \left[J_{i-1}(ke) - J_{i+1}(ke) \right] X_{k,i}^{\ell,m} \right\} +$$

$$+ \sum_{i \geq k-m} \left\{ J_i(ke)\, Z_{k,i}^{\ell,m} + \frac{e\sqrt{1-e^2}}{\beta} \frac{k}{2} \left[J_{i-1}(ke) - J_{i+1}(ke) \right] X_{k,i}^{\ell,m} \right\} \quad , \tag{118}$$

where, for $i < \kappa - m$,

$$Y_{k,i}^{\ell,m} \equiv \frac{\partial X_{k,i}^{\ell,m}}{\partial\beta} = -(k-i-m)\frac{1}{\beta} X_{k,i}^{\ell,m} - 2(-\beta)^{k-i-m+1} \binom{\ell+1-m}{k-i-m} \times$$

$$\times \frac{(k-i-\ell-1)(-m-\ell-1)}{k-i-m+1} F(k-i-\ell,\ -m-\ell,\ k-i-m+2;\beta^2) \tag{119}$$

and, for $i \geq k-m$,

$$Z_{k,i}^{\ell,m} \equiv \frac{\partial X_{k,i}^{\ell,m}}{\partial\beta} = -(-k+i+m)\frac{1}{\beta} X_{k,i}^{\ell,m} - 2(-\beta)^{-k+i+m+1} \binom{\ell+1-m}{-k+i+m} \times$$

$$\times \frac{(-k+i-\ell-1)(m-\ell-1)}{-k+i+m+1} F(-k+i-\ell,\ m-\ell,\ -k+i+m+2;\beta^2) \quad . \tag{120}$$

Finally,

$$N_{\ell pj} \equiv \frac{\partial H_{\ell pj}}{\partial e} = \frac{\beta}{e\sqrt{1-e^2}} Y_{\ell-2p+j}^{\ell,\ell-2p} \quad . \tag{121}$$

Let us consider the definitions

$$D^{+}_{\ell mspqkj} = (\ell-2p)\dot{\omega} + (\ell-2p+j)\dot{M} + m\dot{\Omega} + (\ell-2q)\dot{\omega}_{☾} + (\ell-2q+k)\dot{M}_{☾} + s\dot{\Omega}_{☾} \tag{122}$$

and

$$D^{-}_{\ell mspqkj} = (\ell-2p)\dot{\omega} + (\ell-2p+j)\dot{M} + m\dot{\Omega} - (\ell-2q)\dot{\omega}_{☾} - (\ell-2q+k)\dot{M}_{☾} - s\dot{\Omega}_{☾} \quad . \tag{123}$$

We easily establish that

$$B^{(1)}_{\ell mspqkj} \equiv \int \frac{\partial R_{\ell mspqkj}}{\partial M} dt =$$

$$= R_{\ell mspqkj} \begin{vmatrix} U_{\ell}^{m,s} \rightarrow U_{\ell}^{m,s}(\ell-2p+j)/D^{-}_{\ell mspqkj} \\ U_{\ell}^{m,-s} \rightarrow U_{\ell}^{m,-s}(\ell-2p+j)/D^{+}_{\ell mspqkj} \end{vmatrix} , \tag{124}$$

$$B^{(2)}_{\ell mspqkj} \equiv \int \frac{\partial R_{\ell mspqkj}}{\partial \omega}\, dt =$$

$$= R_{\ell mspqkj} \left\{ \begin{matrix} U^{m,s}_{\ell} \rightarrow U^{m,s}_{\ell} (\ell - 2p)/D^{-}_{\ell mspqkj} \\ U^{m,-s}_{\ell} \rightarrow U^{m,-s}_{\ell} (\ell - 2p)/D^{+}_{\ell mspqkj} \end{matrix} \right\} , \quad (125$$

$$B^{(3)}_{\ell mspqkj} \equiv \int \frac{\partial R_{\ell mspqkj}}{\partial \Omega}\, dt =$$

$$= R_{\ell mspqkj} \left\{ \begin{matrix} U^{m,s}_{\ell} \rightarrow U^{m,s}_{\ell} m/D^{-}_{\ell mspqkj} \\ U^{m,-s}_{\ell} \rightarrow U^{m,-s}_{\ell} m/D^{+}_{\ell mspqkj} \end{matrix} \right\} , \quad (126)$$

$$B^{(4)}_{\ell mspqkj} \equiv \int \frac{\partial R_{\ell mspqkj}}{\partial e}\, dt = \frac{a}{\ell}\, B^{(5)}_{\ell mspqkj} \left\{ H_{\ell pj} \rightarrow N_{\ell pj} \right\} , \quad (127)$$

$$B^{(5)}_{\ell mspqkj} \equiv \int \frac{\partial R_{\ell mspqkj}}{\partial a}\, dt =$$

$$= \frac{\ell}{a}\, R_{\ell mspqkj} \left\{ \begin{matrix} U^{m,-s}_{\ell} \rightarrow U^{m,-s}_{\ell} / D^{+}_{\ell mspqkj} \\ U^{m,s}_{\ell} \rightarrow U^{m,s}_{\ell} / D^{-}_{\ell mspqkj} \\ \cos \rightarrow \sin \end{matrix} \right\} , \quad (128)$$

$$B^{(6)}_{\ell mspqkj} \equiv \int \frac{\partial R_{\ell mspqkj}}{\partial I}\, dt = \frac{a}{\ell}\, B^{(5)}_{\ell mspqkj} \left\{ F_{\ell mp} \rightarrow F^{I}_{\ell mp} \right\} , \quad (129)$$

and

$$\Delta' M_{\ell mspqkj} = -\frac{3}{a^2} N^2_{\leftmoon} a^{2-\ell}_{\leftmoon} \int B^{(1)}_{\ell mspqkj}\, dt =$$

$$= -\frac{3}{a^2} N^2_{\leftmoon} a^{2-\ell}_{\leftmoon} R_{\ell mspqkj} \left\{ \begin{matrix} U^{m,s}_{\ell} \rightarrow U^{m,s}_{\ell} / (D^{-}_{\ell mspqkj})^2 \\ U^{m,-s}_{\ell} \rightarrow U^{m,-s}_{\ell} / (D^{+}_{\ell mspqkj})^2 \\ \cos \rightarrow \sin \end{matrix} \right\} . \quad (130)$$

Finally, the short-period perturbations are given, for all terms for which

$$\ell - 2p + j \neq 0 \quad , \tag{131}$$

by

$$\Delta a_{\ell mspqkj} = N_{☾}^2 \, a_{☾}^{2-\ell} \, \frac{2}{na} \, B^{(1)}_{\ell mspqkj} \quad ,$$

$$\Delta e_{\ell mspqkj} = N_{☾}^2 \, a_{☾}^{2-\ell} \, \frac{\sqrt{1-e^2}}{na^2 e} \left[\sqrt{1-e^2} \, B^{(1)}_{\ell mspqkj} - B^{(2)}_{\ell mpsqkj} \right] \quad ,$$

$$\Delta I_{\ell mspqkj} = N_{☾}^2 \, a_{☾}^{2-\ell} \, \frac{\operatorname{cosec} I}{na^2 \sqrt{1-e^2}} \left[B^{(2)}_{\ell mspqkj} \cos I - B^{(3)}_{\ell mspqkj} \right] \quad ,$$

$$\Delta M_{\ell mspqkj} = -N_{☾}^2 \, a_{☾}^{2-\ell} \, \frac{1}{na^2 e} \left[(1-e^2) \, B^{(4)}_{\ell mspqkj} + 2e \, B^{(5)}_{\ell mspqkj} \right] + \Delta' M_{\ell mspqkj} \quad ,$$

$$\Delta \omega_{\ell mspqkj} = N_{☾}^2 \, a_{☾}^{2-\ell} \, \frac{1}{na^2 e \sqrt{1-e^2}} \left[-e \, B^{(6)}_{\ell mspqkj} \cot I + (1-e^2) \, B^{(4)}_{\ell mspqkj} \right] \quad ,$$

$$\Delta \Omega_{\ell mspqkj} = N_{☾}^2 \, a_{☾}^{2-\ell} \, \frac{\operatorname{cosec} I}{na^2 \sqrt{1-e^2}} \, B^{(6)}_{\ell mspqkj} \quad , \tag{132}$$

which completes the calculations.

11. COMPUTATIONAL PROCEDURE FOR SHORT-PERIOD PERTURBATIONS OF HIGH SATELLITES WITH SMALL ECCENTRICITY

The sequence of calculations to obtain short-period perturbations due to a particular term, $R_{\ell mspqkj}$, of the disturbing function (see (117)) is now given.

Given mean elements a, e, I, $M_☾$, $\omega_☾$, $\Omega_☾$, M, ω, Ω, $\mathcal{E}$, $\dot{M}_☾$, $\dot{\omega}_☾$, $\dot{\Omega}_☾$, $\dot{M}$, $\dot{\omega}$, $\dot{\Omega}$, β, compute, for any term (ℓ, m, s, p, q, k, j), $\ell - 2p + j \neq 0$:

1) $\theta^☾_{\ell sqk}$, $\theta_{\ell mpj}$ (112), (113)

2) $J_i[(\ell - 2p + j)e]$ (Bessel function (34)) to the approximation required

3) $X^{\ell, \ell - 2p}_{\ell - 2p + j, i}(\beta)$ (115) or (116)

4) $F_{\ell mp}(I)$ (9)

5) $F_{\ell sq}(I_☾)$ (9)

6) $G_{\ell qk}(e_☾)$ (33), (34), (35), and (36)

7) $H_{\ell pj}(e)$ (114)

8) $Y^{\ell, m}_{k, i}(\beta)$ (119)

9) $Z^{\ell, m}_{k, i}(\beta)$ (120)

10) $Y^{\ell, m}_{k}(\beta)$ (118)

11) $N_{\ell pj}(\beta)$ (121)

12) $D^{\pm}_{\ell mspqkj}$ (122), (123)

13) $U_{\ell}^{m,s}$, $U_{\ell}^{m,-s}$ (12)

14) $B_{\ell mspqkj}^{(i)}$, $i = 1, 2, \ldots, 6$ (125) through (129)

15) $\Delta' M_{\ell mspqkj}$ (130)

16) $\Delta a_{\ell mspqkj}, \ldots, \Delta\Omega_{\ell mspqkj}$ (132)
Complete short-period perturbations.

12. REMARKS ON SOLAR PERTURBATIONS

The previous formulations apply as well to solar perturbations, which are about of the same order of magnitude. In fact, for the Sun,

$$R = \frac{Gm_\odot}{r_\odot} \sum_{\ell \geq 2} \left(\frac{r}{r_\odot}\right)^\ell P_\ell (\cos \psi'_\odot) \quad , \tag{133}$$

so that

$$Gm_\odot = \frac{m_\odot}{m_☾ + m_\oplus + m_\odot} n_\odot^2 a_\odot^3 \simeq n_\odot^2 a_\odot^3$$

or

$$Gm_\odot = N_\odot^2 a_\odot^3 \quad , \qquad N_\odot^2 \simeq 0.75 \times 10^{-5} \text{ rev}^2 \text{ day}^{-2} \quad ,$$

which is of the same size as $N_☾^2$. For the Sun, we have (to the mean equinox of date):

$$\omega_\odot = 281°13'15''.0 + 6189''.03 \text{ T} + 1''.63 \text{ T}^2 + 0''.012 \text{ T}^3 \quad ,$$

$$M_\odot = 358°28'33''.0 + 129596579''.10 \text{ T} - 0''.54 \text{ T}^2 - 0''.012 \text{ T}^3 \quad ,$$

$$e_\odot = 0.01675104 \text{ (supposed constant)} \quad ,$$

$$a_\odot = 1.00000129 \text{ (astronomical units)} \quad ,$$

$$n_\odot = 3548''.19283 \text{ day}^{-1} \quad .$$

We can consider $I_\odot$, $\Omega_\odot$ to be zero. The mean inclination with respect to the equator is $\mathcal{E}$. For that matter, it cou'd be considered a function of time, but such precision is hardly necessary. The disturbing function is given by Equation (8), while (2) is written

$$R = \sum_{\ell \geq 2} N_\odot^2 \, a_\odot^{2-\ell} \, R_\ell \quad . \tag{134}$$

The transformation (10) is not necessary, so that the coefficients A_ℓ^m, B_ℓ^m (Eq. 5) retain their original form by using $I' = \mathcal{E}$, the inclination of the orbit of the Sun with respect to the equator. It follows that

$$R_\ell = a^\ell \left(\frac{r}{a}\right)^\ell \left(\frac{a_\odot}{r_\odot}\right)^{\ell+1} \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=0}^{\ell} \epsilon_m \frac{(\ell - m)!}{(\ell + m)!} F_{\ell mp}(I) \, F_{\ell mq}(\mathcal{E}) \times$$

$$\times \cos\left[(\ell - 2p) v - (\ell - 2q) v_\odot + m\Omega\right] \quad . \tag{135}$$

The secular and long-period part of this is

$$\overline{R}_\ell = a^\ell \sum_{m=0}^{\ell} \sum_{p=0}^{\ell} \sum_{q=0}^{\ell} \sum_{k=-\infty}^{\infty} \epsilon_m \frac{(\ell - m)!}{(\ell + m)!} F_{\ell mp}(I) \, F_{\ell mq}(\mathcal{E}) \times$$

$$\times (1 + \beta^2)^{-\ell - 1} H_{\ell p(2p - \ell)}(\beta) \, G_{\ell qk}(e_\odot) \times$$

$$\times \cos\left[(\ell - 2p)\omega - (\ell - 2q)\omega_\odot - (\ell - 2q + k) M_\odot + m\Omega\right] \quad , \tag{136}$$

which for the Sun is used in place of Equation (37). More precisely:

$$\overline{R}_{\ell mpqk} = \epsilon_m \frac{(\ell - m)!}{(\ell + m)!} F_{\ell mp}(I) F_{\ell mq}(\varepsilon) (1+\beta^2)^{-\ell - 1} H_{\ell p(2p-\ell)}(\beta) \times$$

$$\times G_{\ell qk}(e_{\odot}) C_{\ell pqmk} , \tag{137}$$

where

$$C_{\ell pqmk} = \cos \left[(\ell - 2p)\,\omega - (\ell - 2q)\omega_{\odot} - (\ell - 2q + k)M_{\odot} + m\Omega\right] . \tag{138}$$

From this point on, all formulas developed for the Moon can be easily adapted, a task not worth undertaking here. The complete expressions are given by Kaula (1962), and the computational procedure is similar to the ones given in the previous sections.

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BIOGRAPHICAL NOTE

GIORGIO E. O. GIACAGLIA received his B.S. in 1958 from the Polytechnical School in São Paulo, Brazil, and his Ph. D. in astronomy from Yale in 1965.

From 1966 to 1970, he was professor of General Mechanics at the University of São Paulo, and from 1970 to 1972, Visiting Professor of Aerospace Engineering at the University of Texas. In 1972, he was appointed Dean of the Institute of Astronomy and Geophysics and Director of the Astronomical Observatory at the University of São Paulo. He is jointly on the scientific staff of the Smithsonian Astrophysical Observatory, specializing in research in celestial mechanics.